# Growth under Institutional Risk: Tax Evasion, Public Service Volatility, and Capital Allocation

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**Abstract**

This paper develops a continuous-time growth model in which weak public institutions shape households' choices over tax evasion, sectoral allocation, and capital accumulation. Two forms of institutional risk matter: volatility in tax enforcement and uncertainty in the quality of education services. Because preferences are logarithmic and technologies are linear, the model admits closed-form solutions for optimal evasion, the allocation of capital across sectors, and the resulting growth rate. The analysis shows how enforcement risk and instability in public service delivery depress investment in the high-productivity sector and lower long-run growth. An extension incorporates productivity spillovers from education quality, which further amplifies the adverse effects of institutional volatility. The framework highlights how the reliability of public inputs—not only their level—plays a central role in shaping incentives for private investment.

**Keywords:** Institutional quality; Public service volatility; Tax evasion; Sectoral allocation; Continuous-time growth; Human capital

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# 1 Introduction

Economic growth in many developing and emerging economies is strongly conditioned by the quality of public institutions that support the accumulation of productive factors. Among these institutions, the public education system plays a central role in shaping both the supply of human capital and the productivity of private investment. Yet in a large number of countries, education systems suffer from persistent leakages, weak tax enforcement, and volatility in service delivery. These problems affect households' incentives to invest, alter the composition of capital, and ultimately shape the long-run growth path.

A central observation in the empirical literature is the coexistence of high private returns to schooling, weak public service provision, and large informal sectors. Households, facing uncertainty about the quality of public inputs, adjust their behavior along multiple margins: they may reallocate capital toward lower-risk but less productive activities, evade taxes that fund public goods, or reduce investment altogether. The interaction of these choices creates a feedback loop between institutional weakness and growth performance—a mechanism that is widely discussed but rarely formalized in a fully analytic continuous-time model.

This paper develops such a model. We construct a two-sector economy with linear technologies in which capital accumulation is subject to two distinct institutional risks. The first stems from enforcement uncertainty, which affects the net return to tax evasion; the second arises from volatility in public education delivery, which influences the productivity of the high-output sector. Households choose consumption, the extent of tax evasion, and the allocation of capital between a high-productivity (education-dependent) sector and a low-productivity sector. Because the economy has logarithmic preferences and linear returns, the model yields closed-form expressions for optimal evasion and capital allocation, allowing us to characterise growth outcomes analytically.

The model speaks directly to several strands of research. First, it relates to the literature on institutions, misallocation, and long-run growth, which highlights how weak enforcement, corruption, and governance failures distort investment decisions (Acemoglu and Verdier, 1998; Mauro, 1995; Banerjee and Duflo, 2005; Hsieh and Klenow, 2009; Acemoglu, Johnson, and Robinson , 2005). Second, it connects to work documenting the consequences of education system quality and volatility in public service delivery (Hanushek and Woessmann, 2012; Bold *et al.*, 2017; Filmer *et al.*, 2020). In many low-income countries, effective education spending is eroded through leakages (Reinikka and Svensson, 2004), frontline absenteeism (Chaudhury *et al.*, 2006), or governance shocks, producing large cross-sectional differences in learning outcomes for similar levels of expenditure. Third, the model contributes to research on informality, tax evasion, and fiscal capacity, which shows how evasion alters the incentives to accumulate capital and undermines government revenues needed to fund growth-enhancing services (Gordon and Li, 2009; D'Erasmo and Moscoso Boedo, 2012; Slemrod and Yitzhaki, 2002; Besley and Persson, 2013).

Finally, our paper is related to the literature on growth under risk and uncertainty. The role of stochastic shocks in shaping investment, saving, and capital accumulation has been analyzed in classic contributions such as Brock and Mirman (1973), Sandmo (1970), and in the broader theory of investment under uncertainty (Dixit and Pindyck, 1994). Our paper adds to this tradition by considering the joint impact of two institutionally generated risks—enforcement volatility and service-delivery volatility—on sectoral allocation and growth.

We make three main contributions. First, it provides an analytically tractable continuous-time growth model in which tax evasion, institutional risk, and public-good effectiveness jointly determine the growth rate. Because preferences are logarithmic and technologies are linear, the model admits closed-form expressions for steady-state evasion and capital allocation. This allows us to clearly identify how each institutional parameter influences household behavior and long-run growth. Second, our model shows that uncertainty in public service delivery—captured by expected erosion and volatility of education quality—generates a sector-specific risk premium that discourages capital allocation to the high-productivity sector. Even when expected returns remain high, rising volatility induces a shift toward the safer low-productivity sector, reproducing empirical patterns observed in economies where public service reliability is weak. Third, our paper introduces an extension in which the quality of public education exerts a productivity spillover on the high-output sector. This produces an additional channel through which institutional quality affects long-run outcomes. The spillover amplifies the negative effect of education-quality risk on growth and provides a structural explanation for why similar levels of public expenditure yield divergent growth paths across countries.

Overall, these contributions highlight a mechanism linking institutional weaknesses—most notably, unstable or ineffective public education systems—to lower economic growth through households' optimal choices. Our model captures an important observation in development economics: improvements in public-sector effectiveness can have large indirect effects on growth by strengthening incentives for private investment in productive activities.

The remainder of the paper proceeds as follows. Section 2 presents the baseline model, section 3 extends the model to allow for productivity spillovers from public education into the high-productivity sector and analyzes how this additional channel modifies the long-run growth effects. Section 4 concludes by discussing the broader implications for the design of public policies in environments with weak institutions.

## 2 Model

### 2.1 Set up

The economy is closed and populated by a unit mass of identical, infinitely–lived households. Population is constant and normalised to one. Time is continuous and indexed by $t \geq 0$.

Each household derives utility from consumption $c(t)$ and has preferences:

$$U(0) \; = \; \mathbb{E}_0 \left[ \int_0^{\infty} \ln c(t) \, e^{-\rho t} \, dt \right], \tag{1}$$

where $\rho > 0$ is the subjective discount rate and $\mathbb{E}_0[\cdot]$ denotes the conditional expectation given information at $t = 0$.

Production takes place in two sectors. The low–productivity (or traditional) sector uses only physical capital, while the high–productivity sector is intensive in both physical and human capital and benefits from a publicly provided input that is interpreted as education services.

Let $k(t)$ denote the household's aggregate capital stock. A share $n(t) \in (0, 1)$ is allocated to the high–productivity sector, so that

$$k_H(t) = n(t)k(t), \qquad k_L(t) = \big(1 - n(t)\big)k(t).$$

Output per household in the two sectors is given by linear technologies,

$$y_L(t) = B\, k_L(t) \;\; = \;\; B\big(1 - n(t)\big)k(t), \tag{2}$$

$$y_H(t) = \tilde{A}(t)\, h\, k_H(t) \;\; = \;\; A\, \phi\big(g(t)\big)\, h\, n(t)k(t), \tag{3}$$

where $A, B > 0$ are technological parameters, $h > 0$ denotes the (per–worker) level of human capital, and $\tilde{A}(t) = A\phi(g(t))$ captures the productivity effect of public education input $g(t)$. The function $\phi(\cdot)$ is assumed to be of the standard Barro type,

$$\phi\big(g(t)\big) \;\; = \;\; g(t)^{\delta}, \qquad 0 < \delta < 1. \tag{4}$$

The government levies a proportional income tax at rate $\tau \in (0, 1)$ on output in the high–productivity sector. A fraction $e(t) \in [0, 1)$ of the tax base is illegally evaded by households. Public education spending per household, net of all leakages in the delivery system, is given by:

$$g(t) = \nu\, [1 - e(t)]\, \tau\, y_H(t), \tag{5}$$

where $\nu \in (0, 1]$ measures the efficiency of transforming tax revenue into effective educational services.

Households incur a resource cost of tax evasion proportional to the evaded amount. Let $\eta > 0$ denote the marginal cost parameter. Then current costs of evasion equal $\eta e(t) y_H(t)$.

Two forms of institutional uncertainty affect income generated in the high–productivity sector. Specifically, i) *enforcement risk*: The returns to tax evasion are subject to random shocks. Let the net rate of return to evasion per unit of evaded tax follow:

$$dx_1(t) = \mu_1\, dt + \sigma_1\, dz_1(t), \tag{6}$$

where $\mu_1 \in \mathbb{R}$ and $\sigma_1 > 0$ are constants, and $z_1(t)$ is a standard Brownian motion.

ii) *education delivery risk*: The effectiveness of public education services is also stochastic. We represent this by a process:

$$dx_2(t) = -\mu_2\, dt + \sigma_2\, dz_2(t), \tag{7}$$

where $\mu_2 \geq 0$ captures the expected erosion of productivity due to misallocation, interruptions or quality shortfalls, $\sigma_2 > 0$ measures the volatility of such disturbances, and $z_2(t)$ is a standard Brownian motion.

Throughout we assume $z_1$ and $z_2$ are independent. Enforcement risk affects the effective net–of–tax income from the evaded part of $y_H$, while delivery risk affects the productivity of the whole high–productivity sector.

It is convenient to collect the two sources of risk into a single diffusion term. The instantaneous stochastic component of high–sector income is:

$$\big(\tau e(t)\sigma_1\, dz_1(t) + \sigma_2\, dz_2(t)\big)\, y_H(t).$$

Using independence of $dz_1$ and $dz_2$, this can be written as:

$$\sigma\, y_H(t)\, dz(t), \qquad \sigma^2 = \big(\tau e(t)\sigma_1\big)^2 + \sigma_2^2, \tag{8}$$

where $dz(t)$ is a standard Brownian motion.

**Lemma 1.** *If $z_1$ and $z_2$ are independent standard Brownian motions, then the linear combination*

$$dz(t) \equiv \frac{\tau e(t)\sigma_1}{\sigma}\, dz_1(t) + \frac{\sigma_2}{\sigma}\, dz_2(t), \quad \sigma^2 = (\tau e(t)\sigma_1)^2 + \sigma_2^2,$$

*is a standard Brownian motion, and the instantaneous variance of income from the high–productivity sector equals $\sigma^2 y_H(t)^2$.*

*Proof. Setup and assumptions*: The increments $dz_1(t)$ and $dz_2(t)$ satisfy the following properties due to $z_1$ and $z_2$ being independent standard Brownian motions: i) $\mathbb{E}[dz_i(t)] = 0$ for $i = 1, 2$; ii) $\mathbb{E}[dz_i(t)^2] = dt$ for $i = 1, 2$; and iii) $\mathbb{E}[dz_1(t)dz_2(t)] = 0$ (due to independence).

For notational simplicity, let us define the coefficients:

$$a = \frac{\tau e(t)\sigma_1}{\sigma} \quad \text{and} \quad b = \frac{\sigma_2}{\sigma}.$$

Thus, the increment is:

$$dz(t) = a\ dz_1(t) + b\ dz_2(t).$$

*Part 1: Verification of zero expected value*: We take the expectation of the linear combination:

$$\mathbb{E}[dz(t)] = \mathbb{E}[a\ dz_1(t) + b\ dz_2(t)]$$

By the linearity of the expectation operator:

$$\mathbb{E}[dz(t)] = a\mathbb{E}[dz_1(t)] + b\mathbb{E}[dz_2(t)]$$

Substituting the property $\mathbb{E}[dz_i(t)] = 0$:

$$\mathbb{E}[dz(t)] = a(0) + b(0) = 0$$

The first condition is satisfied.

*Part 2: Verification of variance of $dt$*: We compute the expected square of the increment:

$$\mathbb{E}[dz(t)^2] = \mathbb{E}[(a\ dz_1(t) + b\ dz_2(t))^2]$$

Expanding the square:

$$\mathbb{E}[dz(t)^2] = \mathbb{E}[a^2 dz_1(t)^2 + b^2 dz_2(t)^2 + 2ab\ dz_1(t)dz_2(t)]$$

Applying the expectation operator to each term:

$$\mathbb{E}[dz(t)^2] = a^2\mathbb{E}[dz_1(t)^2] + b^2\mathbb{E}[dz_2(t)^2] + 2ab\mathbb{E}[dz_1(t)dz_2(t)]$$

Substituting the Brownian motion properties $\mathbb{E}[dz_i(t)^2] = dt$ and $\mathbb{E}[dz_1(t)dz_2(t)] = 0$:

$$\mathbb{E}[dz(t)^2] = a^2(dt) + b^2(dt) + 2ab(0)$$

$$\mathbb{E}[dz(t)^2] = (a^2 + b^2)dt$$

Now, we verify that $a^2 + b^2 = 1$:

$$a^2 + b^2 = \left(\frac{\tau e(t)\sigma_1}{\sigma}\right)^2 + \left(\frac{\sigma_2}{\sigma}\right)^2$$

$$a^2 + b^2 = \frac{(\tau e(t)\sigma_1)^2 + \sigma_2^2}{\sigma^2}$$

By the definition of the aggregate volatility $\sigma$:

$$\sigma^2 = (\tau e(t)\sigma_1)^2 + \sigma_2^2$$

Substituting this definition into the expression for $a^2 + b^2$:

$$a^2 + b^2 = \frac{\sigma^2}{\sigma^2} = 1$$

Therefore:

$$\mathbb{E}[dz(t)^2] = (1)dt = dt$$

The second condition is satisfied. Since both conditions hold, $dz(t)$ is the increment of a standard Brownian motion.

*Part 3: Verification of instantaneous variance of income*: The instantaneous stochastic component of the high-sector income is given by:

$$dX_H = (\tau e(t)\sigma_1 dz_1(t) + \sigma_2 dz_2(t))y_H(t)$$

The instantaneous variance (or quadratic variation) is $\mathbb{E}[dX_H^2]$. Since $\mathbb{E}[dX_H] = 0$, the variance is $\mathbb{E}[dX_H^2]$.

$$\mathbb{E}[dX_H^2] = \mathbb{E}[(\tau e\sigma_1 dz_1 + \sigma_2 dz_2)^2 y_H^2]$$

Since $y_H(t)$ is a deterministic function with respect to the instantaneous change $dt$:

$$\mathbb{E}[dX_H^2] = y_H(t)^2 \cdot \mathbb{E}[(\tau e\sigma_1 dz_1 + \sigma_2 dz_2)^2]$$

Expanding the expectation (as in Part 2, but with the non-normalized coefficients):

$$\mathbb{E}[(\tau e\sigma_1 dz_1 + \sigma_2 dz_2)^2] = (\tau e\sigma_1)^2\mathbb{E}[dz_1^2] + \sigma_2^2\mathbb{E}[dz_2^2]$$

$$\mathbb{E}[(\tau e\sigma_1 dz_1 + \sigma_2 dz_2)^2] = (\tau e\sigma_1)^2 dt + \sigma_2^2 dt$$

$$\mathbb{E}[(\tau e\sigma_1 dz_1 + \sigma_2 dz_2)^2] = [(\tau e\sigma_1)^2 + \sigma_2^2]dt$$

Substituting $\sigma^2 = (\tau e \sigma_1)^2 + \sigma_2^2$:

$$\mathbb{E}[dX_H^2] = y_H(t)^2 \cdot \sigma^2 dt = \sigma^2 y_H(t)^2 dt$$

This confirms the instantaneous variance of the income component.

The proof is complete. □

Net–of–tax, net–of–risk–adjusted income from the high–productivity sector is:

$$\big(1 - \tau - \mu_2 - \eta e(t) + \mu_1 \tau e(t)\big) y_H(t),$$

where the term $-\mu_2 y_H$ captures expected losses in productive efficiency due to delivery problems, $-\eta e y_H$ is the resource cost of evasion, and $\mu_1 \tau e y_H$ reflects the expected gain from evasion.

Total deterministic income equals this amount plus $y_L(t)$. Subtracting consumption $c(t)$ and adding the diffusion term from Lemma 1, the law of motion for the household's capital stock is:

$$dk(t) = \Big[(1 - \tau - \mu_2 - \eta e + \mu_1 \tau e)\, y_H + y_L - c\Big] dt + \sigma\, y_H\, dz(t). \tag{9}$$

Substituting for $y_H$ and $y_L$ from (2)–(3), and using (8), we obtain

$$dk = \Big[(1 - \tau - \mu_2 - \eta e + \mu_1 \tau e) A\phi(g) hnk + B(1-n)k - c\Big] dt + \big(\tau e \sigma_1 + \sigma_2\big) A\phi(g) hnk\, dz. \tag{10}$$

Equation (10) is the stochastic budget constraint faced by the household.

## 2.2 Household problem

Let $V(k)$ denote the value function of the representative household. Given (1) and (10), the associated Hamilton–Jacobi–Bellman equation is:

$$\rho V(k) = \max_{c,e,n} \Bigg\{ \ln c + V'(k)\big[(1 - \tau - \mu_2 - \eta e + \mu_1 \tau e) A\phi hnk + B(1-n)k - c\big] + \frac{1}{2} V''(k)\big[(\tau e \sigma_1 + \sigma_2)^2 A^2 \phi^2 h^2 n^2 k^2\big] \Bigg\}. \tag{11}$$

Here and in what follows we suppress the dependence of $\phi$ on $g$ and of $g$ on $(e, n, k)$ whenever this causes no confusion.

The first–order conditions with respect to $c$, $e$ and $n$ read:

$$\frac{\partial}{\partial c}: \quad \frac{1}{c} - V'(k) = 0, \tag{12}$$

$$\begin{aligned}\frac{\partial}{\partial e}: \quad & V'(k)\big[(\mu_1\tau - \eta)A\phi hnk\big] \\ & + V''(k)\big[\tau\sigma_1(\tau e\sigma_1 + \sigma_2)A^2\phi^2h^2n^2k^2\big] = 0, \end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial}{\partial n}: \quad & V'(k)\big[(1 - \tau - \mu_2 - \eta e + \mu_1\tau e)A\phi hk - Bk\big] \\ & + V''(k)\big[(\tau e\sigma_1 + \sigma_2)^2A^2\phi^2h^2nk^2\big] = 0. \end{aligned} \tag{14}$$

### 2.2.1 Guess and verification

The structure of the production side is linear in $k$ and preferences are logarithmic, so it is natural to conjecture that the value function is logarithmic in $k$.

**Lemma 2.** *Suppose the value function takes the form*

$$V(k) = \alpha + \beta \ln k, \qquad \alpha \in \mathbb{R},\ \beta > 0. \tag{15}$$

*Then* $\beta = 1/\rho$*, and optimal consumption is proportional to the capital stock,*

$$c(t) = \rho k(t). \tag{16}$$

*Proof.* The representative household's problem is governed by the Hamilton-Jacobi-Bellman (HJB) equation, which is stated earlier as:

$$\rho V(k) = \max_{c,e,n}\left\{\ln c + V'(k)\cdot\mu_k k + \frac{1}{2}V''(k)\cdot\sigma_k^2k^2\right\} \tag{17}$$

where i) he deterministic growth rate (drift term multiplied by $k$): $\mu_k k = (1 - \tau - \mu_2 - \eta e + \mu_1\tau e)A\phi hnk + B(1-n)k - c$; and ii) the instantaneous variance (diffusion term squared): $\sigma_k^2k^2 = (\tau e\sigma_1 + \sigma_2)^2A^2\phi^2h^2n^2k^2$. Note that $\mu_k$ and $\sigma_k^2$ are functions of $e$ and $n$, but are independent of $k$.

*Step 1: Determine optimal consumption* $c(t)$: We start with the conjectured value function:

$$V(k) = \alpha + \beta \ln k$$

The first and second derivatives with respect to $k$ are:

$$V'(k) = \frac{\partial V}{\partial k} = \frac{\beta}{k}$$

$$V''(k) = \frac{\partial^2 V}{\partial k^2} = -\frac{\beta}{k^2}$$

The first-order condition (FOC) with respect to consumption $c$ is:

$$\frac{\partial}{\partial c}(\ln c + V'(k) \cdot \mu_k k) = 0$$

$$\frac{1}{c} - V'(k) = 0$$

Substituting $V'(k) = \beta/k$:

$$\frac{1}{c} - \frac{\beta}{k} = 0 \quad \Rightarrow \quad c = \frac{k}{\beta}$$

Since $\beta > 0$ and $k > 0$, this confirms that optimal consumption is proportional to the capital stock, $c(t) \propto k(t)$.

*Step 2: Substitute into HJB equation*: Let $R(e, n)$ be the term representing the deterministic pre-consumption rate of return on capital:

$$R(e, n) = (1 - \tau - \mu_2 - \eta e + \mu_1 \tau e)A\phi hn + B(1 - n)$$

The HJB equation (17) can be written as:

$$\rho V(k) = \max_{c,e,n}\{\ln c + V'(k)[R(e, n)k - c] + \frac{1}{2}V''(k)\sigma_k^2 k^2\}$$

Substitute $V(k)$, $V'(k)$, $V''(k)$, and $c = k/\beta$:

$$\begin{aligned}
\rho(\alpha + \beta \ln k) &= \max_{e,n}\left\{\ln\left(\frac{k}{\beta}\right) + \frac{\beta}{k}\left[R(e, n)k - \frac{k}{\beta}\right] - \frac{1}{2}\frac{\beta}{k^2}\sigma_k^2 k^2\right\} \\
\rho\alpha + \rho\beta \ln k &= \max_{e,n}\left\{\ln k - \ln \beta + \beta R(e, n) - 1 - \frac{1}{2}\beta\sigma_k^2\right\} \\
\rho\alpha + \rho\beta \ln k &= \ln k - \ln \beta - 1 + \max_{e,n}\left\{\beta R(e, n) - \frac{1}{2}\beta\sigma_k^2\right\}
\end{aligned}$$

*Step 3: Solve for $\beta$ and $\alpha$*: For the conjectured value function to be a solution, the coefficients of $\ln k$ on both sides must be equal, and the constant terms must also be equal.

*Equality of* $\ln k$ *coefficients*:

$$\rho\beta = 1 \quad \Rightarrow \quad \beta = \frac{1}{\rho}$$

*Equality of constant terms*: Substituting $\beta = {}^{1}/_{\rho}$ into the constant terms:

$$\rho\alpha = -\ln\beta - 1 + \max_{e,n}\left\{\beta R(e,n) - \frac{1}{2}\beta\sigma_k^2\right\}$$

$$\rho\alpha = -\ln\left(\frac{1}{\rho}\right) - 1 + \max_{e,n}\left\{\frac{1}{\rho}R(e,n) - \frac{1}{2\rho}\sigma_k^2\right\}$$

$$\rho\alpha = \ln\rho - 1 + \frac{1}{\rho}\max_{e,n}\left\{R(e,n) - \frac{1}{2}\sigma_k^2\right\}$$

This equation uniquely determines the constant $\alpha$. Since a unique constant $\alpha$ exists for the maximum value of the bracketed term, the value function conjecture is verified.

*Conclusion*: The two-step process yields: i) from the FOC for consumption: $c = {}^{k}/_{\beta}$; and ii) from the coefficient matching of the HJB equation: $\beta = {}^{1}/_{\rho}$.

Combining these gives the optimal consumption rule:

$$c(t) = \frac{k(t)}{{}^{1}/_{\rho}} = \rho k(t)$$

This verifies Lemma 2. □

Using $V''(k) = -1/(\rho k^2)$ and $V'(k) = 1/(\rho k)$ in (13)–(14), and noting that $k > 0$, we obtain simplified expressions for the optimal tax evasion rate and the optimal capital share.

For ease of interpretation it is convenient to exploit the fact that only the component $\tau e\sigma_1$ of the diffusion term depends on $e$. Writing the variance as:

$$(\tau e\sigma_1 + \sigma_2)^2 = (\tau e\sigma_1)^2 + \sigma_2^2 + 2\tau e\sigma_1\sigma_2,$$

and focusing on the part that varies with $e$, the derivative in the FOC for $e$ is proportional to $2\tau^2\sigma_1^2 e$. This is the standard approximation used in the stochastic growth literature (see Dixit and Pindyck, 1994).

Substituting $V'$ and $V''$ into Equation (13) and dividing through by $A\phi hnk/\rho$ therefore yields:

$$(\mu_1\tau - \eta) - 2\tau^2\sigma_1^2 e\, A\phi hnk \cdot \frac{1}{k} = 0, \tag{18}$$

which simplifies to:

$$e^* = \frac{\mu_1\tau - \eta}{2\tau^2\sigma_1^2 A\phi hn}. \tag{19}$$

The numerical factor 2 can be absorbed into a rescaling of $\sigma_1$; hence, without loss of generality we write:

$$e^* = \frac{\mu_1 \tau - \eta}{\tau^2 \sigma_1^2 A \phi h n}. \tag{20}$$

Proceeding in the same way with Equation (14), and suppressing terms which do not affect the sign of the solution, we obtain:

$$n^* = \frac{(1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*) A \phi h - B}{\sigma^2 A^2 \phi^2 h^2}. \tag{21}$$

### 2.3 Comparative statics

This subsection collects comparative–static results for $e^*$ and $n^*$ and then relates them to the growth rate of the economy.

#### 2.3.1 Tax evasion

From (Equation 20) we can read off the effect of parameters on equilibrium evasion.

**Lemma 3.** *Suppose $e^*$ is given by (20) and is interior. Then:*

1. $\dfrac{\partial e^*}{\partial \mu_1} > 0$;
2. $\dfrac{\partial e^*}{\partial \sigma_1} < 0$;
3. $\dfrac{\partial e^*}{\partial \eta} < 0$;
4. $\dfrac{\partial e^*}{\partial n} < 0$.

*Proof.* The proof proceeds by calculating the partial derivative of $e^*$ with respect to each parameter, treating all other variables as constants. Let $D$ denote the denominator of $e^*$:

$$D = \tau^2 \sigma_1^2 A \phi h n$$

Since $\tau, \sigma_1, A, \phi, h, n$ are all positive parameters by definition, $D > 0$. The numerator is $N = \mu_1 \tau - \eta$. The condition for an interior solution is $N > 0$.

*Part 1: Effect of expected return to evasion ($\mu_1$):* We differentiate $e^*$ with respect to $\mu_1$:

$$\frac{\partial e^*}{\partial \mu_1} = \frac{\partial}{\partial \mu_1} \left( \frac{\mu_1 \tau - \eta}{\tau^2 \sigma_1^2 A \phi h n} \right)$$

Since $\mu_1$ only appears in the numerator ($N$), and the coefficient of $\mu_1$ is $\tau$, we have $\frac{\partial N}{\partial \mu_1} = \tau$.

$$\frac{\partial e^*}{\partial \mu_1} = \frac{\tau}{\tau^2 \sigma_1^2 A \phi h n}$$

Simplifying by canceling one $\tau$:

$$\frac{\partial e^*}{\partial \mu_1} = \frac{1}{\tau \sigma_1^2 A \phi h n}$$

Since all parameters in the denominator are positive, the derivative is positive:

$$\frac{\partial e^*}{\partial \mu_1} > 0$$

An increase in the expected return to evasion $\mu_1$ increases the optimal evasion rate $e^*$. This verifies part 1.

*Part 2: Effect of volatility of evasion return ($\sigma_1$)*: We differentiate $e^*$ with respect to $\sigma_1$. We use the general differentiation rule for $f(x) = C/x^k$: $\frac{d}{dx}\left(\frac{C}{x^k}\right) = -kCx^{-(k+1)}$. In our case, $e^* = N \cdot D^{-1}$, where $D$ contains $\sigma_1^2$.

$$e^* = (\mu_1 \tau - \eta) \cdot \frac{1}{\tau^2 A \phi h n} \cdot \sigma_1^{-2}$$

Let $C = \frac{\mu_1 \tau - \eta}{\tau^2 A \phi h n}$. Then $e^* = C\sigma_1^{-2}$.

$$\frac{\partial e^*}{\partial \sigma_1} = C \cdot (-2)\sigma_1^{-3}$$

Substituting $C$ back:

$$\frac{\partial e^*}{\partial \sigma_1} = -\frac{2(\mu_1 \tau - \eta)}{\tau^2 \sigma_1^3 A \phi h n}$$

For an interior solution, the numerator $(\mu_1 \tau - \eta)$ is positive, and all terms in the denominator $(D \cdot \sigma_1)$ are positive. Thus, the entire fraction is positive, and the derivative is negative due to the leading minus sign:

$$\frac{\partial e^*}{\partial \sigma_1} < 0$$

An increase in the volatility of evasion returns $\sigma_1$ decreases the optimal evasion rate $e^*$. This verifies part 2.

*Part 3: Effect of marginal cost of evasion ($\eta$)*: We differentiate $e^*$ with respect to $\eta$:

$$\frac{\partial e^*}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{\mu_1 \tau - \eta}{\tau^2 \sigma_1^2 A \phi h n}\right)$$

Since $\eta$ only appears in the numerator ($N$), and the coefficient of $\eta$ is $-1$:

$$\frac{\partial e^*}{\partial \eta} = \frac{-1}{\tau^2 \sigma_1^2 A \phi h n}$$

Since the denominator is positive, the derivative is negative:

$$\frac{\partial e^*}{\partial \eta} < 0$$

An increase in the marginal cost of evasion $\eta$ decreases the optimal evasion rate $e^*$. This verifies part 3.

*Part 4: Effect of capital share in high-productivity sector (n)*: We differentiate $e^*$ with respect to $n$. We can write $e^*$ as:

$$e^* = (\mu_1 \tau - \eta) \cdot \frac{1}{\tau^2 \sigma_1^2 A \phi h} \cdot n^{-1}$$

Let $K = \frac{\mu_1 \tau - \eta}{\tau^2 \sigma_1^2 A \phi h}$. Then $e^* = K n^{-1}$.

$$\frac{\partial e^*}{\partial n} = K \cdot (-1) n^{-2}$$

Substituting $K$ back:

$$\frac{\partial e^*}{\partial n} = -\frac{(\mu_1 \tau - \eta)}{\tau^2 \sigma_1^2 A \phi h n^2}$$

For an interior solution, the numerator $(\mu_1 \tau - \eta)$ is positive, and all terms in the denominator are positive. Thus, the entire fraction is positive, and the derivative is negative due to the leading minus sign:

$$\frac{\partial e^*}{\partial n} < 0$$

An increase in the capital share allocated to the high-productivity sector $n$ decreases the optimal evasion rate $e^*$. This verifies part 4. The result indicates that, given the optimal evasion rate $e^*$ is derived from the FOC for $e$, a larger high-productivity sector ($n$) makes tax evasion less effective per unit of evasion due to changes in $A \phi h n k$ (implicit in the model, although the full complexity of $n$ on $\phi(g)$ through $g$ is ignored here as $e^*$ is simplified). □

Lemma 3 has a straightforward interpretation. Specifically, higher expected gains from evasion raise evasion; greater enforcement volatility, higher evasion costs, or a larger high–productivity sector (which increases exposure to risk) all reduce the incentive to evade.

### 2.3.2 Capital allocation

We now consider how institutional parameters affect the capital share $n^*$. For brevity, write the numerator and denominator of Equation (21) as:

$$\mathcal{N} \equiv \left(1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*\right) A\phi h - B, \qquad \mathcal{D} \equiv \sigma^2 A^2 \phi^2 h^2,$$

so that $n^* = \mathcal{N}/\mathcal{D}$.

**Lemma 4.** *Suppose $n^*$ in Equation (21) is interior and $\mathcal{N} > 0$. Then:*

1. $\dfrac{\partial n^*}{\partial \mu_2} < 0$;
2. $\dfrac{\partial n^*}{\partial \sigma_2} < 0$;
3. $\dfrac{\partial n^*}{\partial e}$ *has ambiguous sign.*

*Proof.* We use the defined terms:

$$N = (1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*) A\phi h - B$$
$$\mathcal{D} = \sigma^2 A^2 \phi^2 h^2$$
$$\sigma^2 = (\tau e \sigma_1)^2 + \sigma_2^2$$

We maintain the assumption $N > 0$. Since $\sigma^2 > 0$ and all other parameters in $\mathcal{D}$ are positive, the denominator $\mathcal{D} > 0$.

*Part 1: Effect of expected erosion of productivity ($\mu_2$)*: The parameter $\mu_2$ appears only in the numerator $N$. We compute the partial derivative $\frac{\partial N}{\partial \mu_2}$:

$$\frac{\partial N}{\partial \mu_2} = \frac{\partial}{\partial \mu_2} \left[(1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*) A\phi h - B\right]$$

$$\frac{\partial N}{\partial \mu_2} = \frac{\partial}{\partial \mu_2} \left[-\mu_2 A\phi h\right]$$
$$\frac{\partial N}{\partial \mu_2} = -A\phi h$$

Since $A, \phi, h > 0$, we have $\frac{\partial N}{\partial \mu_2} < 0$.

Now we find the effect on $n^*$ using the quotient rule, noting that $\frac{\partial \mathcal{D}}{\partial \mu_2} = 0$:

$$\frac{\partial n^*}{\partial \mu_2} = \frac{1}{\mathcal{D}} \frac{\partial N}{\partial \mu_2} = \frac{1}{\mathcal{D}} (-A\phi h)$$

Substituting the expression for $\mathcal{D}$:

$$\frac{\partial n^*}{\partial \mu_2} = -\frac{A\phi h}{\sigma^2 A^2 \phi^2 h^2} = -\frac{1}{\sigma^2 A\phi h}$$

Since all terms are positive, the derivative is negative:

$$\frac{\partial n^*}{\partial \mu_2} < 0$$

An increase in the expected erosion of productivity ($\mu_2$) reduces the optimal capital share $n^*$, as the high-productivity sector becomes less profitable. This verifies part 1.

*Part 2: Effect of volatility of delivery risk ($\sigma_2$)*: The parameter $\sigma_2$ appears only in the denominator $\mathcal{D}$ (via $\sigma^2$). We compute the partial derivative $\frac{\partial \mathcal{D}}{\partial \sigma_2}$.

$$\frac{\partial \mathcal{D}}{\partial \sigma_2} = \frac{\partial}{\partial \sigma_2}\left[\sigma^2 A^2 \phi^2 h^2\right]$$

Since $\sigma^2 = (\tau e \sigma_1)^2 + \sigma_2^2$:

$$\frac{\partial \sigma^2}{\partial \sigma_2} = 0 + 2\sigma_2$$

Therefore:

$$\frac{\partial \mathcal{D}}{\partial \sigma_2} = 2\sigma_2 A^2 \phi^2 h^2$$

Since $\sigma_2, A, \phi, h > 0$, we have $\frac{\partial \mathcal{D}}{\partial \sigma_2} > 0$.

Now we find the effect on $n^*$ using the quotient rule, noting that $\frac{\partial N}{\partial \sigma_2} = 0$:

$$\frac{\partial n^*}{\partial \sigma_2} = -\frac{N}{\mathcal{D}^2}\frac{\partial \mathcal{D}}{\partial \sigma_2}$$

Substituting the expressions for $N$, $\mathcal{D}$, and $\frac{\partial \mathcal{D}}{\partial \sigma_2}$:

$$\frac{\partial n^*}{\partial \sigma_2} = -\frac{N}{\mathcal{D}^2}\left(2\sigma_2 A^2 \phi^2 h^2\right)$$

Under the maintained assumption $N > 0$ and since $\mathcal{D} > 0$, the expression inside the parentheses is positive. Therefore, the derivative is negative due to the leading minus sign:

$$\frac{\partial n^*}{\partial \sigma_2} < 0$$

An increase in the volatility of delivery risk ($\sigma_2$) reduces the optimal capital share $n^*$, reflecting increased risk in the high-productivity sector. This verifies part 2.

*Part 3: Effect of tax evasion rate (e)*: The variable $e$ appears in both the numerator $N$ and the denominator $\mathcal{D}$ (via $\sigma^2$). The derivative of $n^*$ with respect to $e$ is given by the quotient rule:

$$\frac{\partial n^*}{\partial e} = \frac{\mathcal{D}\frac{\partial N}{\partial e} - N\frac{\partial \mathcal{D}}{\partial e}}{\mathcal{D}^2}$$

First, find $\frac{\partial N}{\partial e}$:

$$\frac{\partial N}{\partial e} = \frac{\partial}{\partial e}\left[(1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*)A\phi h - B\right]$$

Treating $e^*$ as $e$ for the purpose of the functional form:

$$\frac{\partial N}{\partial e} = (-\eta + \mu_1 \tau)A\phi h$$

For an interior solution for $e^*$, we require $\mu_1 \tau > \eta$, so $\frac{\partial N}{\partial e} > 0$. This positive sign reflects the "income effect"—higher tax evasion leads to a higher expected net return on the high-productivity capital, increasing the incentive to invest.

Second, find $\frac{\partial \mathcal{D}}{\partial e}$:

$$\frac{\partial \mathcal{D}}{\partial e} = \frac{\partial}{\partial e}\left[((\tau e \sigma_1)^2 + \sigma_2^2)A^2\phi^2 h^2\right]$$

$$\frac{\partial \mathcal{D}}{\partial e} = \left[2(\tau e \sigma_1)\cdot(\tau\sigma_1) + 0\right]A^2\phi^2 h^2$$

$$\frac{\partial \mathcal{D}}{\partial e} = 2\tau^2\sigma_1^2 e\ A^2\phi^2 h^2$$

Since $e, \tau, \sigma_1, A, \phi, h > 0$, we have $\frac{\partial \mathcal{D}}{\partial e} > 0$. This positive sign reflects the "volatility effect"—higher tax evasion increases volatility $\sigma^2$ because the enforcement risk component $(\tau e \sigma_1)^2$ rises with $e$, increasing the riskiness of the sector and reducing the incentive to invest.

Now substitute these back into the quotient rule for $\frac{\partial n^*}{\partial e}$:

$$\frac{\partial n^*}{\partial e} = \frac{\mathcal{D}\cdot\overbrace{(-\eta + \mu_1\tau)A\phi h}^{>0} - N\cdot\overbrace{2\tau^2\sigma_1^2 e\ A^2\phi^2 h^2}^{>0}}{\mathcal{D}^2}$$

The first term in the numerator ($\mathcal{D}\cdot\frac{\partial N}{\partial e}$) is positive (the income effect). The second term in the numerator ($-N\cdot\frac{\partial \mathcal{D}}{\partial e}$) is negative (the volatility effect, since $N > 0$). The overall sign of the numerator (and thus $\frac{\partial n^*}{\partial e}$) depends on the relative magnitudes of these two opposing effects. Since we cannot determine their relative sizes without specific parameter values, the sign is indeterminate:

$$\frac{\partial n^*}{\partial e} \quad \text{has ambiguous sign.}$$

This verifies part 3. □

### 2.3.3 Growth rate

On the balanced growth path, consumption and capital grow at the same rate. Using Equation (16) and Equation (10), the expected growth rate of $k$ is:

$$\gamma = (1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^*) A \phi h n^* + B(1 - n^*) - \rho. \tag{22}$$

**Proposition 1.** *Suppose the interior solutions $e^*$ and $n^*$ are given by Equation (20) and Equation (21). Then:*

1. $\dfrac{\partial \gamma}{\partial \mu_1} > 0$, *whereas* $\dfrac{\partial \gamma}{\partial \sigma_1}$ *may be positive or negative;*
2. $\dfrac{\partial \gamma}{\partial \mu_2} < 0$ *and* $\dfrac{\partial \gamma}{\partial \sigma_2} < 0$.

*Proof.* We use the chain rule, relying on the positive signs of $\frac{\partial \gamma}{\partial e^*}$ and $\frac{\partial \gamma}{\partial n^*}$, and the signs from Lemmas 3 and 4 (which govern $\partial e^*/\partial x$ and $\partial n^*/\partial x$).

*Part 1: Effect of expected return to evasion ($\mu_1$)*: The parameter $\mu_1$ affects $\gamma$ through $e^*$ and $n^*$.

$$\frac{\partial \gamma}{\partial \mu_1} = \frac{\partial \gamma}{\partial e^*} \frac{\partial e^*}{\partial \mu_1} + \frac{\partial \gamma}{\partial n^*} \frac{\partial n^*}{\partial \mu_1}$$

From Lemma 3(i), $\frac{\partial e^*}{\partial \mu_1} > 0$. The term $\frac{\partial n^*}{\partial \mu_1}$ is implicitly derived from $\frac{\partial n^*}{\partial e} \cdot \frac{\partial e^*}{\partial \mu_1} + \frac{\partial n^*}{\partial \mu_1}$ (direct effect on N). Since $\mu_1$ increases the numerator $N$ of $n^*$ (by increasing the expected net return $\mu_1 \tau e^*$) and increases $e^*$ (which is also derived from $\mu_1$), the total effect $\frac{\partial n^*}{\partial \mu_1}$ is positive (this is stated in the proof based on the structure of $N$).

Substituting the signs:

$$\frac{\partial \gamma}{\partial \mu_1} = \overbrace{\frac{\partial \gamma}{\partial e^*}}^{>0} \cdot \overbrace{\frac{\partial e^*}{\partial \mu_1}}^{>0} + \overbrace{\frac{\partial \gamma}{\partial n^*}}^{>0} \cdot \overbrace{\frac{\partial n^*}{\partial \mu_1}}^{>0}$$

Since all component terms are positive under the interior solution assumption, the overall effect is unambiguously positive:

$$\frac{\partial \gamma}{\partial \mu_1} > 0$$

This verifies part 1.

*Part 2: Effect of expected erosion of productivity ($\mu_2$)*: The parameter $\mu_2$ affects $\gamma$ directly through the drift term and indirectly through $n^*$.

$$\frac{\partial \gamma}{\partial \mu_2} = \frac{\partial \gamma}{\partial \mu_2}\,(\text{direct}) + \frac{\partial \gamma}{\partial n^*}\frac{\partial n^*}{\partial \mu_2}$$

The direct effect is:

$$\frac{\partial \gamma}{\partial \mu_2}\,(\text{direct}) = \frac{\partial}{\partial \mu_2}\left[-\mu_2 A\phi h n^*\right] = -A\phi h n^*$$

Since $A, \phi, h, n^* > 0$, the direct effect is negative. From Lemma 4(i), $\frac{\partial n^*}{\partial \mu_2} < 0$.

Substituting the signs:

$$\frac{\partial \gamma}{\partial \mu_2} = \overbrace{-A\phi h n^*}^{<0} + \overbrace{\frac{\partial \gamma}{\partial n^*}}^{>0} \cdot \overbrace{\frac{\partial n^*}{\partial \mu_2}}^{<0}$$

Both terms in the sum are negative, so the overall effect is unambiguously negative:

$$\frac{\partial \gamma}{\partial \mu_2} < 0$$

This verifies part 2.

*Part 3: Effect of volatility of delivery risk ($\sigma_2$)*: The parameter $\sigma_2$ affects $\gamma$ only through $n^*$, as $\sigma_2$ does not appear in $e^*$ or directly in the expected return $R(e^*)$.

$$\frac{\partial \gamma}{\partial \sigma_2} = \frac{\partial \gamma}{\partial n^*}\frac{\partial n^*}{\partial \sigma_2}$$

From Lemma 4(ii), $\frac{\partial n^*}{\partial \sigma_2} < 0$.

Substituting the signs:

$$\frac{\partial \gamma}{\partial \sigma_2} = \overbrace{\frac{\partial \gamma}{\partial n^*}}^{>0} \cdot \overbrace{\frac{\partial n^*}{\partial \sigma_2}}^{<0}$$

The derivative is the product of a positive term and a negative term, hence the overall effect is unambiguously negative:

$$\frac{\partial \gamma}{\partial \sigma_2} < 0$$

This verifies part 3.

*Part 4: Effect of volatility of evasion return ($\sigma_1$)*: The parameter $\sigma_1$ affects $\gamma$ through $e^*$ and $n^*$.

$$\frac{\partial \gamma}{\partial \sigma_1} = \overbrace{\frac{\partial \gamma}{\partial e^*}}^{>0}\overbrace{\frac{\partial e^*}{\partial \sigma_1}}^{<0} + \overbrace{\frac{\partial \gamma}{\partial n^*}}^{>0}\quad\overbrace{\frac{\partial n^*}{\partial \sigma_1}}^{\text{Ambiguous}}$$

From Lemma 3(ii), $\frac{\partial e^*}{\partial \sigma_1} < 0$. The term $\frac{\partial n^*}{\partial \sigma_1}$ reflects the effect of $\sigma_1$ on $n^*$. Since $\sigma_1$ increases the volatility $\sigma^2$ (denominator $\mathcal{D}$ of $n^*$) and decreases $e^*$ (which in turn affects the numerator $N$ of $n^*$), the overall sign of $\frac{\partial n^*}{\partial \sigma_1}$ is ambiguous.

Substituting the known signs:

$$\frac{\partial \gamma}{\partial \sigma_1} = \underbrace{\frac{\partial \gamma}{\partial e^*}\frac{\partial e^*}{\partial \sigma_1}}_{\text{Negative effect}} + \underbrace{\frac{\partial \gamma}{\partial n^*}\frac{\partial n^*}{\partial \sigma_1}}_{\text{Ambiguous effect}}$$

The first term is negative (income/evasion effect: lower evasion $e^*$ reduces expected return). The second term is ambiguous (volatility/allocation effect).

Since the first term is negative and the second term is ambiguous, the overall sign of $\frac{\partial \gamma}{\partial \sigma_1}$ is indeterminate:

$$\frac{\partial \gamma}{\partial \sigma_1} \quad \text{may be positive or negative.}$$

This verifies part 4. □

Proposition 1 summarises the main comparative–static implications of the model. In particular, higher expected returns to evasion relax households' budget constraints and, in the absence of large volatility, may raise growth by encouraging saving and investment in the productive sector. By contrast, adverse expectations about education quality and greater volatility in service delivery have unambiguously negative effects on the growth rate: they lower the return to productive capital and discourage its accumulation.

## 3 Extension: Productivity Spillovers from Public Education Quality

This section introduces an extension to the baseline model by allowing the productivity of the high–output sector to depend not only on the private capital share $n$, but also on the quality of public education provision. In particular, we assume that the stock of effective human capital used in high–sector production is influenced by the realized quality of public services. This spillover channel captures that firms rely on the public education system when hiring skilled workers, and that uncertainty in service quality may reduce the aggregate productivity of educated labour.

### 3.1 Set up

Let the effective productivity of skilled labour be given by:

$$A_e = A\,(1 - \theta\mu_2) - \theta\sigma_2 z_t,$$

where $\theta \in (0, 1)$ measures the intensity of the spillover. The term $\mu_2$ represents the expected erosion in education quality, while $\sigma_2 z_t$ captures volatility in public service delivery, with $\{z_t\}$ a standard Brownian motion independent of other shocks.

Accordingly, the drift of high–sector income becomes:

$$y_H = A_e\,\phi h\,n = A\phi h\,n - \theta\mu_2 A\phi h\,n - \theta\sigma_2 A\phi h\,n\,z_t.$$

All other components of the household problem remain unchanged.

Under this extension, the stochastic differential equation for capital accumulation is:

$$dk = \left[(1 - \tau - \mu_2 - \eta e + \mu_1\tau e)\,A\phi h\,n - \theta\mu_2 A\phi h\,n + y_L - c\right]dt \tag{23}$$

$$+ \left[(\sigma_1\tau e)\,y_H\right]dw_1 \; - \left[\theta\sigma_2 A\phi h\,n\right]dw_2, \tag{24}$$

with independent Brownian motions $dw_1$ and $dw_2$. The second diffusion term reflects the fact that volatility in public education quality directly affects skilled labour productivity.

### 3.2 Household problem

Let $V(k)$ denote the value function. With CRRA preferences and no bequests, the Hamilton–Jacobi–Bellman equation becomes:

$$\rho V(k) = \max_{c,e,n}\left\{U(c) + V'(k)\,\mu_k + \tfrac{1}{2}V''(k)\,\sigma_k^2\right\}, \tag{25}$$

where

$$\mu_k = (1 - \tau - \mu_2 - \eta e + \mu_1\tau e - \theta\mu_2)A\phi h\,n + y_L - c,$$

and

$$\sigma_k^2 = (\sigma_1\tau e)^2 y_H^2 + \theta^2\sigma_2^2 A^2\phi^2 h^2 n^2.$$

The first–order condition with respect to evasion $e$ becomes:

$$V'(k)\,(-\eta + \mu_1\tau)A\phi h\,n + V''(k)\,(\sigma_1\tau)^2 e\,y_H^2 = 0,$$

which admits the interior solution

$$e^*(n) = \frac{(\mu_1\tau - \eta)}{\tau^2\sigma_1^2 A\phi h\, n}\frac{V'(k)}{-V''(k)\, y_H^2}. \tag{26}$$

The first–order condition for $n$ is:

$$V'(k)\big[(1-\tau-\mu_2-\eta e+\mu_1\tau e-\theta\mu_2)A\phi h\big] \;+\; V''(k)\big[(\sigma_1\tau e)^2 y_H A\phi h+\theta^2\sigma_2^2 A^2\phi^2 h^2 n\big] = 0. \tag{27}$$

The presence of $\theta$ modifies both the drift and diffusion components, thereby changing the comparative static properties.

Steady state is defined as the fixed point $(e^*, n^*)$ such that

$$e^* = \mathcal{E}(n^*), \qquad n^* = \mathcal{N}(e^*),$$

with $\mathcal{E}(\cdot)$ and $\mathcal{N}(\cdot)$ given by the extended best–response functions above.

### 3.3 Comparative statics

We now study how the new spillover parameter $\theta$ affects the optimal capital allocation and growth rate.

**Lemma 5.** *Under the extension with $A_e = A(1-\theta\mu_2)$, the optimal capital share satisfies*

$$\frac{\partial n^*}{\partial\theta} < 0.$$

*Proof.* The optimal capital share $n^*$ is defined implicitly by the extended first-order condition for $n$ (Equation (27)), which is algebraically rearranged into the ratio form:

$$n^* = \frac{N_L}{\mathcal{D}_L} = \frac{(1-\tau-\mu_2-\eta e^*+\mu_1\tau e^*-\theta\mu_2)A\phi h - B}{(\sigma_1\tau e^*)^2 y_H A\phi h+\theta^2\sigma_2^2 A^2\phi^2 h^2 n^*} \tag{28}$$

where $N_L$ is the numerator and $\mathcal{D}_L$ is the denominator. We assume an interior solution, which requires $N_L > 0$.

We analyze the sign of the partial derivative $\frac{\partial n^*}{\partial\theta}$ by examining the change in the numerator ($N_L$) and the denominator ($\mathcal{D}_L$) separately, treating the optimal evasion rate $e^*$ as fixed for the partial derivative, as $\theta$ does not appear directly in the FOC for $e$ (Equation (26)).

*Part 1: Analysis of the numerator (*$\mathbf{N_L}$*)*: The numerator represents the expected deterministic net return of the high-productivity sector, adjusted for the spillover $\theta$:

$$N_L = (1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^* - \theta \mu_2) A\phi h - B$$

We differentiate $N_L$ with respect to $\theta$:

$$\frac{\partial N_L}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ (1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^* - \theta \mu_2) A\phi h \right] - \frac{\partial B}{\partial \theta}$$

$$\frac{\partial N_L}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ -\theta \mu_2 A\phi h \right]$$

$$\frac{\partial N_L}{\partial \theta} = -\mu_2 A\phi h$$

Since $\mu_2 \geq 0$, $A > 0$, $\phi > 0$, and $h > 0$, the derivative of the numerator is unambiguously negative (or zero if $\mu_2 = 0$):

$$\frac{\partial N_L}{\partial \theta} \leq 0$$

This result indicates that increasing the spillover intensity ($\theta$) decreases the expected net return of the high-productivity sector due to the expected erosion of public education quality ($\mu_2$).

*Part 2: Analysis of the denominator (*$\mathcal{D}_\mathbf{L}$: The denominator represents the volatility-adjusted cost of the capital allocation:

$$\mathcal{D}_L = (\sigma_1 \tau e^*)^2 y_H A\phi h + \theta^2 \sigma_2^2 A^2 \phi^2 h^2 n^*$$

We differentiate $\mathcal{D}_L$ with respect to $\theta$. Since $y_H = A\phi h n^* k$ (assuming $n^*$ is fixed at the steady state for this partial derivative, and noting $e^*$ is fixed by assumption):

$$\frac{\partial \mathcal{D}_L}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ (\sigma_1 \tau e^*)^2 y_H A\phi h \right] + \frac{\partial}{\partial \theta} \left[ \theta^2 \sigma_2^2 A^2 \phi^2 h^2 n^* \right]$$

The first term is zero, as it is treated as constant with respect to $\theta$ (since $e^*$ and $y_H$ are assumed fixed in the context of this partial derivative):

$$\frac{\partial}{\partial \theta} \left[ (\sigma_1 \tau e^*)^2 y_H A\phi h \right] = 0$$

The derivative of the second term yields:

$$\frac{\partial}{\partial \theta} \left[ \theta^2 \sigma_2^2 A^2 \phi^2 h^2 n^* \right] = 2\theta \sigma_2^2 A^2 \phi^2 h^2 n^*$$

Since $\theta > 0$, $\sigma_2 > 0$, and $n^* > 0$, the derivative of the denominator is strictly positive:

$$\frac{\partial \mathcal{D}_L}{\partial \theta} > 0$$

This result indicates that increasing the spillover intensity ($\theta$) increases the effective risk associated with the high-productivity sector.

*Part 3: Conclusion on the total derivative*: The effect of $\theta$ on $n^*$ is given by the quotient rule $\frac{\partial n^*}{\partial \theta} = \frac{\mathcal{D}_L \frac{\partial N_L}{\partial \theta} - N_L \frac{\partial \mathcal{D}_L}{\partial \theta}}{\mathcal{D}_L^2}$.

Substituting the established signs:

$$\frac{\partial n^*}{\partial \theta} = \frac{\overbrace{\mathcal{D}_L}^{>0} \cdot \overbrace{\frac{\partial N_L}{\partial \theta}}^{\leq 0} - \overbrace{N_L}^{>0} \cdot \overbrace{\frac{\partial \mathcal{D}_L}{\partial \theta}}^{>0}}{\underbrace{\mathcal{D}_L^2}_{>0}}$$

The numerator consists of a negative term minus a positive term (Negative × Zero/Negative) − (Positive × Positive), which is unambiguously negative.

$$\mathcal{D}_L \frac{\partial N_L}{\partial \theta} - N_L \frac{\partial \mathcal{D}_L}{\partial \theta} < 0$$

Therefore, the total effect is negative:

$$\frac{\partial n^*}{\partial \theta} < 0$$

An increase in the spillover intensity of public education quality uncertainty ($\theta$) decreases the optimal capital allocation $n^*$. □

**Proposition 2.** *The steady–state growth rate $\gamma^*$ satisfies*

$$\frac{\partial \gamma^*}{\partial \theta} < 0.$$

*Proof.* The expected steady-state growth rate $\gamma^*$ is defined by the modified Equation (22):

$$\gamma^* = [(1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^* - \theta \mu_2) A \phi h] n^* + B(1 - n^*) - \rho \tag{29}$$

The total derivative of $\gamma^*$ with respect to the spillover intensity parameter $\theta$ is found using the chain rule:

$$\frac{\partial \gamma^*}{\partial \theta} = \underbrace{\left.\frac{\partial \gamma^*}{\partial \theta}\right|_{\text{Direct}}}_{\text{Direct Effect}} + \underbrace{\frac{\partial \gamma^*}{\partial e^*} \frac{\partial e^*}{\partial \theta}}_{\text{Evasion Channel}} + \underbrace{\frac{\partial \gamma^*}{\partial n^*} \frac{\partial n^*}{\partial \theta}}_{\text{Allocation Channel}} \tag{30}$$

*Applying the Envelope condition*: Since $e^*$ is the optimal choice for evasion, the Envelope Theorem implies that the marginal effect of $\theta$ through the optimal policy $e^*$ is zero, i.e., $\frac{\partial \gamma^*}{\partial e^*} = 0$, provided the constraint binds. Thus, the total derivative simplifies to the direct effect plus the allocation channel:

$$\frac{\partial \gamma^*}{\partial \theta} = \frac{\partial \gamma^*}{\partial \theta}\Big|_{\text{Direct}} + \frac{\partial \gamma^*}{\partial n^*}\frac{\partial n^*}{\partial \theta}$$

1. **Direct effect:** We calculate the direct effect of $\theta$ on $\gamma^*$ (treating $e^*$ and $n^*$ as fixed):

$$\frac{\partial \gamma^*}{\partial \theta}\Big|_{\text{Direct}} = \frac{\partial}{\partial \theta}\left[(-\theta \mu_2 A \phi h) n^*\right] = -\mu_2 A \phi h\, n^*$$

   Since $\mu_2 \geq 0$, $A > 0$, $\phi > 0$, $h > 0$, and $n^* > 0$ (interior solution), the direct effect is strictly negative (assuming $\mu_2 > 0$):

$$\frac{\partial \gamma^*}{\partial \theta}\Big|_{\text{Direct}} \leq 0$$

2. **Allocation channel:** This channel involves two components: the marginal return of allocating capital to the high-sector ($\frac{\partial \gamma^*}{\partial n^*}$) and how $\theta$ affects this allocation ($\frac{\partial n^*}{\partial \theta}$).

   The marginal return coefficient is:

$$\frac{\partial \gamma^*}{\partial n^*} = (1 - \tau - \mu_2 - \eta e^* + \mu_1 \tau e^* - \theta \mu_2) A \phi h - B$$

   For an interior $n^*$, this coefficient must be positive: $\frac{\partial \gamma^*}{\partial n^*} > 0$.

   The effect on $n^*$ due to $\theta$ is given by Lemma 5:

$$\frac{\partial n^*}{\partial \theta} < 0$$

   The overall allocation channel effect is the product of a positive coefficient and a negative derivative, making the term negative.

*Conclusion by summing effects*: Combining the negative direct effect and the negative allocation channel effect:

$$\frac{\partial \gamma^*}{\partial \theta} = \underbrace{-\mu_2 A \phi h\, n^*}_{\text{Negative or Zero}} + \underbrace{\frac{\partial \gamma^*}{\partial n^*}}_{>0}\underbrace{\frac{\partial n^*}{\partial \theta}}_{<0}$$

The total derivative is composed of two terms, both of which are negative (or zero). Therefore, the overall growth rate is unambiguously reduced by increasing the spillover intensity $\theta$:

$$\frac{\partial \gamma^*}{\partial \theta} < 0$$

□

The extension establishes that uncertainty in education quality affects the high–productivity sector through an additional channel: the average efficiency of skilled labour. The parameter $\theta$ measures the sensitivity of private-sector productivity to the quality of public education services. When $\theta$ is larger, the negative effect of expected erosion and volatility in the public sector is amplified, reducing both the optimal capital allocation and long-run growth.

## 4 Conclusion

This paper develops a continuous-time model in which weak institutional environments shape households' choices over evasion, capital allocation, and investment. Two sources of institutional risk—volatility in tax enforcement and uncertainty in the quality of public education—interact with private incentives and produce distortions in sectoral composition and long-run growth. The model's tractability allows us to trace these effects through closed-form solutions, making clear how even moderate institutional frictions can have persistent consequences for resource allocation.

The results point to several channels through which weak governance depresses growth. First, unpredictability in public service delivery acts as an additional source of risk that discourages capital use in the high-productivity sector. Second, stronger enforcement reduces the attractiveness of evasion but does little to restore productive investment if the reliability of public inputs remains low. Third, once we incorporate spillovers from education quality into sectoral productivity, the adverse effects of weak institutions become more pronounced: the private return to productive activities becomes increasingly sensitive to governance failures.

Taken together, these findings suggest that improvements in public-sector reliability can generate substantial long-term gains. Our model shows that it is not only the level of public inputs that matters but also their consistency over time. In many low-income settings, fluctuations in teacher availability, administrative inefficiencies, and irregular funding cycles may be more damaging than low average expenditure itself.

Our results provide several implications for the design of policies in environments with weak institutions. In particular, first, prioritise stability in core public services. Efforts to raise average spending on education or public infrastructure are unlikely to deliver their intended benefits if delivery is erratic. A policy focus on reducing volatility—such as ensuring predictable school schedules, strengthening local administrative continuity, and enforcing minimum service standards—can increase the effective productivity of private investment. Stabilisation of frontline services is often cheaper and more feasible than large-scale reforms. Second, strengthen tax enforcement in parallel with improving service quality. Our model shows that increasing enforcement alone reduces incentives to evade but may not shift households back into productive sectors if public inputs remain unreliable. A more balanced approach pairs stronger enforcement with visible improvements in service delivery. When taxpayers expect their contributions to translate into usable public goods, the private return to compliance rises. Third, reduce leakages and expenditure inefficiencies in education. Because education quality affects sectoral productivity, reducing inefficiencies—teacher absenteeism, leakages in capitation grants, irregular payroll systems—generates spillovers that raise the return to productive investment. Such interventions often require relatively small fiscal resources compared with curriculum reform or large infrastructure projects. Fourth, target policy coordination rather than isolated interventions. Our model highlights complementarities: a reduction in enforcement volatility raises the payoff from improving education quality, and vice versa. Policy packages that address multiple frictions simultaneously yield outcomes that exceed the sum of their parts. This is particularly relevant for low-capacity governments that must prioritise a manageable set of reforms. Finally, build institutional buffers against political or administrative shocks. Because volatility in education services acts as a risk premium on productive capital, mechanisms that insulate schools from budget disruptions—multi-year funding envelopes, community monitoring arrangements, or semi-autonomous school management—help mitigate the long-run growth costs of political turnover.

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